On the Generalized Enveloping Algebra of a Color Lie Algebra

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Abstract

Let G be an abelien group, ϵ an anti-bicharacter of G and L a G-graded ε Lie algebra (color Lie algebra) over $\mathbb{K}$ a field of characteristic zero. We prove that all G-graded, positive filtered A such that the associated graded algebra is isomorphic to the G-graded ε-symmetric algebra $S(L)$, there is a G- graded ε-Lie algebra L and a G-graded scalar two cocycle $\omega \in \mathrm{Z}^2_{gr}(L, \mathbb{K})$, such that A is isomorphic to $U_\omega(L)$ the generalized enveloping algebra of L associated with ω. We also prove there is an isomorphism of graded spaces between the Hochschild cohomology of the generalized universal enveloping algebra $U(L)$ and the generalized cohomology of color Lie algebra L.

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Introduction

Let G be an abelein group, ϵ an anti-bicharacter of G and $(L, [,])$ a G-graded ε Lie algebra (color Lie algebra) over $\mathbb{K}$ a field of characteristic zero. Let $\omega \in \mathrm{Z}^2_{gr}(L, \mathbb{K})$ be a scalar graded two cocycle of degree zero in the sense of Scheunert-Zhang, [6]. The generalized enveloping algebra (or ω-enveloping algebra) of L is the quotient of the G-graded tensor algebra $T(L)$ by the G-graded two-sided ideal generated by the elements $v_1 \otimes v_2 - \varepsilon(|v_1|, |v_2|) v_2 \otimes v_2 - [v_1, v_2] - \omega(v_1, v_2)$, where v_1, v_2 are homogeneous elements of L. The object of the present paper is to study the structure of the generalized enveloping algebra. In Section 1 we

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fix notation and provide background material concerning finite group gradings and color Lie algebras. In Section 2 we introduce the generalized enveloping algebra of color Lie algebra and study its properties. In particular we state the generalized Poincaré-Birkhoff-Witt theorem for the generalized enveloping algebra. In Section 3 we classify all G-graded, positive filtered A such that the associated graded algebra is isomorphic to the G-graded ε-symmetric algebra $S(L)$ which extends the result of Sridharan for Lie algebras, [7]. In Section 4 we introduce a graded generalized cohomology (or ω-cohomology) of color Lie algebras which coincides with the graded Chevalley-Eilenberg cohomology of degree zero of L introduced by Scheunert and Zhang [6] in the case $\omega = 0$. We show that there is an isomorphism of graded spaces between the Hochschild cohomology of the generalized universal enveloping algebra and the graded ω-cohomology of color Lie algebra.

1 Premilinaries

Throughout this paper groups are assumed to be abelian and $\mathbb{K}$ is a field of characteristic zero. We recall some notation for graded algebras and graded modules [1], and some facts on color Lie algebras from ([5],[6]).

1.1 Graded Hochschild cohomology

Let G be a group with identity element e. We will write G as an multiplicative group. An associative algebra A with unit 1_A, is said to be G-graded, if there is a family $\{A_g | g \in G\}$ of subspaces of A such that $A = \oplus_{g\in G} A_g$ with $1_A \in A_0$ and $A_g A_h \subseteq A_{gh}$, for all $g, h \in G$. Any element $a \in A_g$ is called homogeneous of degree g, and we write $|a| = g$.
A left graded A-module M is a left A-module with a decomposition $M = \oplus_{g\in G} M_g$ such that $A_g.M_h \subseteq M_{gh}$. Let M and N be graded A-modules. Define

$$\mathrm{Hom}_{A\text{-gr}}(M,N) = \{f \in \mathrm{Hom}_A(M,N)| \ \ f(M_g) \subset N_g, \forall \ \ g \in G\}. \tag{1.1}$$

We obtain the category of graded left A-modules, denoted by A-gr, ,[1]. Denote by $\mathrm{Ext}^n_{A\text{-gr}}(-,-)$ the n-th right derived functor of the functor $\mathrm{Hom}_{A\text{-gr}}(-,-)$.
Let us recall the notion of graded Hochschild cohomology of a graded algebra A.
A graded A-bimodule is an A-bimodule $M = \oplus_{g\in G} M_g$ such that $A_g.M_h.A_k \subseteq M_{ghk}$. Thus we obtain the category of graded A-bimodules, denoted by A-A-gr.
Let $A^e = A \otimes A^{op}$ be the enveloping algebra of A, where A^{op} is the opposite algebra of A. The algebra A^e also is graded by G by setting $A^e_g := \sum_{h\in G} A_h \otimes A_{h^{-1}g}$. Now the graded A-bimodule M becomes a graded left A^e-module by defining the A^e-action as

$$(a \otimes b)m = a.m.b, \tag{1.2}$$

and it is clear that $A^e_g M_h \subseteq M_{gh}$, i.e., M is a graded A^e-module. Moreover, every graded left A^e-module arises in this way. Precisely, the above correspondence establishes an equivalence of categories

$$A\text{-}A\text{-gr} \simeq A^e\text{-gr}. \tag{1.3}$$

In the sequel we will identify these categories. Let M be a graded A-bimodule, as above, M may be regarded as a graded left A^e-module. The n-th graded Hochschild cohomology of A with value in M is defined by

$$\mathrm{HH}^n_{\mathrm{gr}}(A, M) := \mathrm{Ext}^n_{A^e\text{-}\mathrm{gr}}(A, M), \quad n \geq 0, \tag{1.4}$$

where A is the graded left A^e-module induced by the multiplication of A, and the algebra $A^e = \oplus_{g \in G} A^e_g$ is considered as a G-graded algebra.

1.2 Lie color algebras

The concept of color Lie algebras is related to an abelian group G and an anti-symmetric bicharacter $\varepsilon : G \times G \to \mathbb{K}^{\times}$, i.e.,

$$\varepsilon(g,h)\,\varepsilon(h,g) = 1, \tag{1.5}$$

$$\varepsilon(g,hk) = \varepsilon(g,h)\,\varepsilon(g,k), \tag{1.6}$$

$$\varepsilon(gh,k) = \varepsilon(g,k)\,\varepsilon(h,k), \tag{1.7}$$

where $g, h, k \in G$ and $\mathbb{K}^{\times}$ is the multiplicative group of the units in $\mathbb{K}$.
A G-graded space $L = \oplus_{g \in G} L_g$ is said to be a G-graded ε-Lie algebra (or simply, color Lie algebra), if it is endowed with a bilinear bracket $[-,-]$ satisfying the following conditions

$$[L_g, L_h] \subseteq L_{gh}, \tag{1.8}$$

$$[a,b] = -\varepsilon(|a|,|b|)\,[b,a], \tag{1.9}$$

$$\varepsilon(|c|,|a|)\,[a,[b,c]] + \varepsilon(|a|,|b|)\,[b,[c,a]] + \varepsilon(|b|,|c|)\,[c,[a,b]] = 0, \tag{1.10}$$

where $g, h \in G$, and $a, b, c \in L$ are homogeneous elements.
For example, a super Lie algebra is exactly a $\mathbb{Z}_2$-graded ε-Lie algebra where

$$\varepsilon(i,j) = (-1)^{ij}, \forall \quad i, j \in \mathbb{Z}_2. \tag{1.11}$$

Let L be a color Lie algebra as above and $T(L)$ the tensor algebra of the G-graded vector space L. It is well-known that $T(L)$ has a natural $\mathbb{Z} \times G$-grading which is fixed by the condition that the degree of a tensor $a_1 \otimes ... \otimes a_n$ with $a_i \in L_{g_i}$, $g_i \in G$, for $1 \leq i \leq n$, is equal to $(n, g_1 + ... + g_n)$. The subspace of $T(L)$ spanned by homogeneous tensors of order $\leq n$ will be denoted by $T^n(L)$.
Let $J(L)$ be the G-graded two-sided ideal of $T(L)$ which is generated by

$$a \otimes b - \varepsilon(|a|,|b|)\, b \otimes a - [a,b] \tag{1.12}$$

with homogeneous $a, b \in \mathfrak{g}$. The quotient algebra $U(L) := T(L)/J(L)$ is called the universal enveloping algebra of the color Lie algebra L. The $\mathbb{K}$-algebra $U(L)$ is a G-graded algebra and has a positive filtration by putting $U_n(L)$ equal to the canonical image of $T_n(L)$ in $T(L)$.

In particular, if L is ε-commutative (i.e., $[L,L] = 0$), then $U(L) = S(L)$ (the ε-symmetric algebra of the graded space L).

The canonical map $\mathrm{i}_L : L \to U(L)$ is a G-graded homomorphism and satisfies

$$i_L(a)\, i_L(b) - \varepsilon(|a|, |b|)\, i_L(b)\, i_L(a) = i_L([a, b]). \qquad (1.13)$$

The $\mathbb{Z}$-graded algebra $G(L)$ associated with the filtered algebra $U(L)$ is defined by letting $G^n(L)$ be the vector space $U_n(L)/U_{n-1}(L)$ and $G(L)$ the space $\oplus_{n\in\mathbb{N}} G^n(L)$ (note $U^{-1}(L) := \{0\}$). Consequently, $G(L)$ is a $\mathbb{Z} \times G$-graded algebra. The well-known generalized Poincaré-Birkhoff-Witt theorem, [5], states that the canonical homomorphism $\mathrm{i}_L : L \to U(L)$ is an injective G-graded homomorphism; moreover, if $\{x_i\}_I$ is a homogeneous basis of L, where the index set I well-ordered. Set $y_{k_j} := \mathrm{i}\left(x_{k_j}\right)$, then the set of ordered monomials $y_{k_1} \cdots y_{k_n}$ is a basis of $U(L)$, where $k_j \leq k_{j+1}$ and $k_j < k_{j+1}$ if $\epsilon(g_j, g_j) \neq 1$ with $x_{k_j} \in L_{g_j}$ for all $1 \leq j \leq n, n \in \mathbb{N}$. In case L is finite-dimensional $U(L)$ is a graded two-sided Noetherian algebra (e.g., see for example [3]).

2 Generalized Enveloping Algebras

Let L be a ϵ-Lie algebra over $\mathbb{K}$, $U(L)$ its enveloping algebra and $S(L)$ its ϵ symmetric algebra. Let $\omega \in \mathrm{Z}^2_{gr}(L, \mathbb{K})$ be a 2-cocycle (of degree zero) for L with values in $\mathbb{K}$ considered as a G-graded trivial L-module, i.e.

$$\epsilon(|z|, |x|)\omega(x, [y, z]) + \epsilon(|x|, |y|)\omega(y, [z, x]) + \epsilon(|y|, |z|)\omega(z, [x, y]) = 0 \qquad (2.1)$$

for all homogeneous elements $x, y, z \in L$, see [6], [2].

Definition 2.1 *Let L be a ϵ-Lie algebra and $\omega \in \mathrm{Z}^2_{gr}(L, \mathbb{K})$ a scalar graded 2-cocycle. We call generalized enveloping algebra of L associated with ω, the algebra $U_\omega(L)$, quotient of the tensor algebra over L by the G-graded two sided ideal generated by the elements of the form $v_1 \otimes v_2 - \epsilon(v_1, v_2) v_2 \otimes v_1 - [v_1, v_2] - \omega(v_1, v_2)$, where v_1, v_2 are homogeneous elements. Then the algebra $U_\omega(L)$ is G-graded and $\mathbb{Z}$-filtered.*

Let $\omega \in \mathrm{Z}^2(L, \mathbb{K})$ be a scalar graded two cocycle of the color Lie algebra L. Let $L_\omega := L \ltimes \mathbb{K} \cdot x$ be a central extension of L with ω such that the new bracket $[,]'$ is defined by

$$[x_1 + ax, x_2 + bx]' := [x_1, x_2] + \omega(x_1, x_2)\, x \qquad (2.2)$$

where $x_1, x_2 \in L, a, b, x \in \mathbb{K}$ are homogeneous. The generalized enveloping algebra $U_\omega(L)$ is isomorphic to the G-graded and $\mathbb{Z}$-filtered algebra $U(L_\omega) / < y - 1 >$, with $< y - 1 >$ being the G-graded two-sided ideal of $U(L_\omega)$ generated by $y - 1$ and y the image of x in $U(L_\omega)$. Denote by

$$\pi_\omega : U(L_\omega) \to U_\omega(L) \qquad (2.3)$$

the canonical epimorphism.

Definition 2.2 *A graded (left) (ω, L)-module over $\mathbb{K}$ is a graded $\mathbb{K}$-module M endowed with a graded $\mathbb{K}$-linear map $\varphi : L \to \mathrm{Hom}_{\mathrm{gr}}(M, M)$ such that for all*

homogeneous elements $x, y \in L$

$$[[\varphi(x), \varphi(y)]] = \varphi([x, y]) + \omega(x, y) i_M \tag{2.4}$$

where $[[\varphi(x), \varphi(y)]] = \varphi(x)\varphi(y) - \varepsilon(|x|, |y|)\varphi(y)\varphi(x)$ *and* i_M *is the graded identity map of* M.

Proposition 2.1 *There is a* $1-1$ *correspondence between graded (left)* (ω, L)*-modules and graded (left)* $U_\omega(L)$ *modules.*

Proof. Let (M, φ) be a graded left (ω, L)-module, then the graded $\mathbb{K}$ linear map φ may be uniquely extended to a graded $\mathbb{K}$ homomorphism $\widehat{\varphi} : T(L) \to \mathrm{Hom}_{\mathrm{gr}}(M, M)$. It follows from the condition (2.4) that $\widehat{\varphi}$ vanishes on the G-graded two sided ideal of $U_\omega(L)$ generated by the elements

$$v_1 \otimes v_2 - \epsilon(|v|_1, |v|_2) v_2 \otimes v_1 - [v_1, v_2] - \omega(v_1, v_2),$$

where v_1, v_2 are homogeneous elements. The converse is trivial. $\square$

This proves in particular that for any $\omega \in \mathrm{Z}^2_{gr}(L, \mathbb{K})$ there is a (ω, L)-module, we can see for example that the ω-enveloping algebra $U_\omega(L)$ is a graded (ω, L)-module.

Theorem 2.1 *If* L *is a* $\mathbb{K}$*-free* ϵ*-Lie algebra. Let* $\{x_i\}_{i \in I}$ *be a* G*-homogeneous basis of* L*, where* I *is a well-ordered set. For any central extension of* L *with* ω*, the set of ordered monomials* $z_{i_1} \cdots z_{i_n}$ *forms a basis of* $U_\omega(L)$*, where* $i_j \leq i_{j+1}$ *and* $i_j < i_{j+1}$ *if* $\epsilon(g_j, g_j) \neq 1$ *with* $y_{i_j} \in L_{g_j}$ *for all* $1 \leq j \leq n, n \in \mathbb{N}$.

Proof. Since $\{x_i\}_{i \in I}$ is a G-homogeneous basis of the vector space L, it follows that $\{x_i, x\}_{i \in I}$ forms a G-homogeneous basis of the vector space L_ω. Let

$$i_\omega : L_\omega \xrightarrow{i_{L_\omega}} U(L_\omega) \xrightarrow{\pi_\omega} U_\omega(L) \tag{2.5}$$

denote the composition. We set $z_i := i_\omega(x_i)$, $z := i_\omega(x)$, $y_i := i_{L_\omega}(x_i)$, with $i \in I$. Let $y^{i_0} y_{i_1} \cdots y_{i_n}$ be the generators of the PBW basis of $U(L_\omega)$ with $i_0 \in \mathbb{N}$, $i_0 \leq i_1$ and $i_0 < i_1$ if $\epsilon(|y_{i_0}|, |y_{i_1}|) \neq 1$. In the quotient algebra $U_\omega(L) = U(L_\omega) / < y - 1 >$, the element z^{i_0} is identified with 1. Then the canonical projection π_ω sends $y^{i_0} y_{i_1} \cdots y_{i_n}$ into $z_{i_1} \cdots z_{i_n}$, and it follows that the elements $z_{i_1} \cdots z_{i_n}$ form a basis of $U_\omega(L)$. $\square$

The restriction of the canonical homomorphism i_ω on L, see (2.5), we is again denoted by i_ω, i.e., $i_\omega : L \to U_\omega(L)$ satisfies for every $x, y \in L$, homogeneous elements:

$$[[i_\omega(x), i_\omega(y)]] = i_\omega([x, y]) + \omega(x, y) \cdot i_{U_\omega(L)} \tag{2.6}$$

with $[[i_\omega(x), i_\omega(y)]] = i_\omega(x) \cdot i_\omega(y) - \epsilon(|x|, |y|) i_\omega(y) \cdot i_\omega(x)$.

Corollary 2.1 *If* L *is a* $\mathbb{K}$*-free* ϵ*-Lie algebra, then for any central extension of* L *with* ω*,* $i_\omega : L \to U_\omega(L)$ *is an injective homomorphism.*

Thus we may identify every element of L with the canonical image in $U_\omega(L)$. Hence L is embedded in $U_\omega(L)$ and

$$[[x,y]] = [x,y] + w(x,y)\cdot 1 \tag{2.7}$$

for all $x, y \in L$. The algebra $U_\omega(L)$ has a positive filtration defined by taking for $U_{n,\omega}(L)$ the canonical image of $U_n(L_\omega)$ by π_ω. Denote by $G_\omega(L)$ its associated $\mathbb{Z}$-graded algebra , then $G_\omega(L)$ is a $\mathbb{Z}\times\Gamma$-graded algebra and ϵ-commutative. It follows that the canonical injection

$$L \overset{i_\omega}{\to} U_\omega(L) \to G_\omega(L), \tag{2.8}$$

may be uniquely extended to a homorphism φ_ω of the ϵ-symmetric algebra $S(L)$ of L into $G_\omega(L)$. If $S^n(L)$ denotes the set of elements of $S(L)$ which are homogeneous of degree $(n, g_1 + ... + g_n)$, then $\varphi_\omega(S^n(L)) \subset G^n_\omega(L)$.

Proposition 2.2 *The canonical homomorphism φ_ω of $S(L)$ into $G_\omega(L)$ is a $\mathbb{Z}\times\Gamma$-graded algebra isomorphism.*

Proof. Let $\{x_i\}_{i\in I}$ be a G-homogeneous basis of L, with I a well-ordered set. Let $y_{i_1}\cdots y_{i_n}$ be the product $x_{i_1}\cdots x_{i_n}$ calculated in $S(L)$, $z_{i_1}\cdots z_{i_n}$ the product $x_{i_1}\cdots x_{i_n}$ calculated in $U_\omega(L)$ and $z'_{i_1}\cdots z'_{i_n}$ the canonical image of $z_{i_1}\cdots z_{i_n}$ in $G_\omega(L)$. Since the set of ordered monomials $z_{i_1}\cdots z_{i_n}$ form a basis of $U_\omega(L)$, by Theorem 2.1, then the set of ordered monomials $z'_{i_1}\cdots z'_{i_n}$ is a basis of $G_\omega(L)$. Since $\varphi_\omega(y_{i_1}\cdots y_{i_n}) = z'_{i_1}\cdots z'_{i_n}$, it can be seen that φ_ω is bijective. □

Proposition 2.3 *If L is of finite dimensional then $U_\omega(L)$ is a graded Noetherian algebra.*

Proof. By Proposition 2.2, the generalized enveloping algebra $U_\omega(L)$ is a positively graded filtered algebra with its associated graded algebra $gr(U_\omega(L)) \simeq S(L)$. The fact that the ϵ-symmetric algebra $S(L)$ is graded Noetherian, see Lemma 2.3 [3] and by Theorem 1.1.9 [4] we deduce that $U_\omega(L)$ is a graded Noetherian algebra. □

3 Classification of Generalized Enveloping Algebras

Fix G an abelien group and ϵ an antisymmetric bicharacter on G. Let V be a free G-graded vector space over $\mathbb{K}$. Let $S(V)$ denote the ϵ-symmetric algebra of V. Consider the family of all pairs $(\mathrm{A}, \varphi_\mathrm{A})$ where $\mathrm{A} = \cup_{n\in\mathbb{Z}_+} F_n\mathrm{A}$ is a G-graded, $\mathbb{Z}$-filtered algebra and $\varphi_\mathrm{A} : S(V) \to G_F(\mathrm{A})$ is a $G\times\mathbb{Z}$-graded isomorphism. A map $\Psi : (A, \varphi_\mathrm{A}) \to (\mathrm{B}, \varphi_\mathrm{B})$ is a G-graded, $\mathbb{Z}$-filtered algebra homomorphism $\Psi : \mathrm{A} \to \mathrm{B}$ such that if $G(\Psi) : G(\mathrm{A}) \to G(\mathrm{B})$ is the $G\times\mathbb{Z}$-graded algebra

morphism induced by Ψ, the diagram

$$\begin{tikzcd} G(\mathrm{A}) \arrow[r, "G(\Psi)"] & G(B) \\ & S(V) \arrow[ul, "\varphi_{\mathrm{A}}"'] \arrow[u, "\varphi_{\mathrm{B}}"'] \end{tikzcd} \tag{3.1}$$

is commutative. Composition of maps is defined in the obvious way. The resulting category is denoted by $\mathfrak{R}_{gr}\left(S(V)\right)$. If $\Psi : (A, \varphi_A) \to (B, \varphi_B)$ is a map then $G\left(\Psi\right) : G\left(A\right) \to G\left(B\right)$ is a graded isomorphism, since $G\left(\Psi\right) = \varphi_{\mathrm{B}} \circ \varphi_{\mathrm{A}}^{-1}$.

Lemma 3.1 *With notation as above $\Psi : \mathrm{A} \to \mathrm{B}$ is a $\mathbb{Z}$-filtered, G-graded isomorphism.*

Proof. Let $\Psi_p : F_p\mathrm{A} \to F_p\mathrm{B}$ denote the $\mathbb{K}$ linear map induced by Ψ. We reason by induction on the integer p. It is clear that Ψ_0 is a graded isomorphism. From the commutativity of the diagram

$$\begin{tikzcd} 0 \arrow[r] & F_{p-1}\mathrm{A} \arrow[r] \arrow[d, "\Psi_{p-1}"] & F_p\mathrm{A} \arrow[r] \arrow[d, "\Psi_p"] & G_p(\mathrm{A}) \arrow[r] \arrow[d, "G_p(\Psi)"] & 0 \\ 0 \arrow[r] & F_{p-1}\mathrm{B} \arrow[r] & F_p\mathrm{B} \arrow[r] & G_p(\mathrm{B}) \arrow[r] & 0 \end{tikzcd} \tag{3.2}$$

at Ψ_{p-1} and $G_p\left(\Psi\right)$ are graded isomorphisms, it is easily seen that Ψ_p is also a graded isomorphism. Since p is arbitrary, the assertion holds. $\square$

Lemma 3.2 *For each (A, φ_A) pair of $\mathfrak{R}(S(V))$ there is a pair $(L, [\omega])$ where L is a ϵ-Lie algebra and $[\omega] \in \mathrm{H}^2_{gr}\left(L, \mathbb{K}\right)$ such that $F_1\mathrm{A} = L_\omega = L \ltimes \mathbb{K}$, with ω is a representative of $[\omega]$.*

Proof. Let $a, b \in F_1A$ be homogeneous elements, we have $[a, b] := ab - \epsilon\left(a, b\right) ba \in F_2A$. Since $G\left(A\right)$ is ϵ-commutative via φ_A, then $[a, b] \in F_1A$. Thus F_1A acquires a structure of a ϵ-Lie algebra. It is clear that $\mathbb{K} = F_0A$ is a central G-graded ideal of F_1A. The G-graded isomorphism $S_1(V) \cong F_1A/F_0A$ given by φ_A, induces a ϵ-Lie structure on $S_1(V)$, denote it by L. Then the following sequence

$$0 \to \mathbb{K} \xrightarrow{\mathrm{i}} F_1A \xrightarrow{\pi} L \to 0 \tag{3.3}$$

is central G-graded exact and π induced by φ_{A}. Thus i and π are graded homomorphisms (of degree zero) of ϵ-Lie algebras. Since $S_1(V)$ is $\mathbb{K}$-free, there exists a graded linear map $\sigma : L \to F_1A$ (necessarily of degree zero) such that $\pi \circ \epsilon = \mathrm{id}_{F_1A}$. We then have

$$\pi([[\sigma(x), \sigma(y)]] - [x, y]) = 0$$

for all (homogeneous) $x, y \in F_1A$. Hence, there is a unique map $\omega : L \times L \to \mathbb{K}$ such that

$$\mathrm{i}(\omega(x, y)) = [[\sigma(x), \sigma(y)]] - \sigma([x, y]) \tag{3.4}$$

for all (homogeneous) $x, y \in L$, and it is easy to see that ω is a homogeneous 2-cocycle of degree zero, i.e, $\omega \in Z^2_{gr}\left(L, \mathbb{K}\right)$. From [6], it follows that the cohomology class $[\omega]$ of ω is independent of the choice of ω. $\square$

Theorem 3.1 *Let G be an abelien group and ϵ a symmetric bicharacter on G. Let V be G-graded $\mathbb{K}$-free module. Let $S(V)$ be the ϵ-symmetric algebra on V. The isomorphism classes of objects in $\mathfrak{R}_{gr}(S(V))$ are in a 1-1 correspondence with pairs $(L,[\omega])$ where L is a ϵ-Lie algebra on V and $[\omega]$ is an element in $\mathrm{H}^2_{gr}(L,\mathbb{K})$. If ω is a cocycle in the cohomology class $[\omega]$, then $(U_\omega(L),\varphi_\omega)$ is an object in the isomorphism class determined by $(L,[\omega])$.*

Proof. Let L be a ϵ-Lie algebra structure on V and ω is a representative of the cohomology class $[\omega]\in \mathrm{H}^2_{gr}(L,\mathbb{K})$. Using Proposition 2.2, then $(U_\omega(L),\varphi_\omega)$ is an object in $\mathfrak{R}_{gr}(S)$. Consider the exact sequence of graded algebras

$$0\to\mathbb{K}\xrightarrow{\mathrm{i}}F_1(U_\omega(L))\overset{\pi_\omega}{\rightrightarrows}L\to 0 \tag{3.5}$$

where π_ω is induced by φ_ω. The map $i_\omega: L\to F_1(U_\omega(L))$ is a $\mathbb{K}$-homogeneous linear section and the relation (2.6) shows that $(U_\omega(L),\varphi_\omega)$ yields $(L,[\omega])$. Let $(A,\varphi_A)\in\mathfrak{R}_{gr}(S(V))$ be another object. Choose $\sigma: L\to F_1A$ so that (3.4) is valid for the cocycle ω. Let $\widehat{\sigma}: T(L)\to A$ be the natural homogeneous extension of σ. If $x,y\in L$ are homogeneous, then,

$$\widehat{\sigma}(x\otimes y-\epsilon(|x|,|y|)y\otimes x-[x,y]-\omega(x,y))=[[\sigma(x),\sigma(y)]]-\sigma([x,y])-\omega(x,y)=0.$$

Then $\widehat{\sigma}$ induces a G-graded, $\mathbb{Z}$-filtered homomorphism of algebras $\overline{\sigma}: U_\omega(L)\to \mathrm{A}$. We then have

$$\begin{tikzcd} G(U_\omega) \arrow[r, "G(\overline{\sigma})"] & G(\mathrm{A}) \\ & S(V) \arrow[ul, "\varphi_\omega"] \arrow[u, "\varphi_{\mathrm{A}}"'] \end{tikzcd} \tag{3.6}$$

For $x\in\mathfrak{g}$, $\sigma(x)$ is in the coset $\varphi_{\mathrm{A}}(x)$ of $F_1\mathrm{A}$ mod $F_0\mathrm{A}$. Thus, $G(\overline{\sigma})\varphi_\omega(x)=G(\overline{\sigma})i_\omega(x)=\varphi_{\mathrm{A}}(x)$. Hence the diagram above is commutative. Thus, $\overline{\sigma}:(U_\omega,\varphi_\omega)\to(\mathrm{A},\varphi_{\mathrm{A}})$ is a map and then an isomorphism by Lemma 3.1. $\square$

From Theorem 3.1 we retain in particular that $(U_{\omega_1}(L),\varphi_{\omega_1})$ and $(U_{\omega_2}(L),\varphi_{\omega_2})$ are $\mathbb{Z}$-filtered, G-graded isomorphic if and only if, ω_1 and ω_2 are (graded) cohomologous.

4 Homological Properties of $\mathcal{U}_\omega(L)$ and Color Hopf Algebra

Let G be a commutative group and $\chi: G\to\mathbb{K}^*$ a bicharacter.

Definition 4.1 *A (G,χ)-Hopf graded algebra A is a 5-tuple $(A,m,\eta,\Delta,\epsilon,S)$ such that*

1. *$A=\oplus_{g\in G}A_g$ is a graded algebra with multiplication $m: A\otimes A\longrightarrow A$ and the unit map $\eta: K\longrightarrow A$. Moreover, (A,Δ,ϵ) is a graded coalgebra with respect to the same grading.*

2. *The counit $\epsilon : A \longrightarrow K$ is an algebra map. The comultiplication $\Delta : A \longrightarrow (A \otimes A)^{\chi}$ is an algebra map, where the algebra $(A \otimes A)^{\chi}$ is equipped with multiplication $*$ defined by*

$$(a \otimes b) * (a' \otimes b') = \chi(|b|, |a'|) aa' \otimes bb', \tag{4.1}$$

where $a, a' \in A$ and $b, b' \in B$ are homogeneous.

3. *The antipode $S : A \longrightarrow A$ is a graded map such that*

$$\sum a_1 S(a_2) = \epsilon(a) = \sum S(a_1) a_2 \tag{4.2}$$

for all homogeneous $a \in A$, where we use Sweedler's notation

$$\Delta(a) = \sum a_1 \otimes a_2$$

.

Definition 4.2 *An algebra is said to be a color Hopf algebra if it is a (G, χ)-Hopf algebra with the antipode being an isomorphism.*

Let M be a graded A-bimodule, then we define a left A-module by

$$am = \sum \chi(|a_{(2)}|, |m|) a_{(1)}.m.S(a_{(2)}), \tag{4.3}$$

for homogeneous $a \in A$ and $m \in M$. It is called the adjoint A-graded module and denoted by ${}^{ad}M$.

Theorem 4.1 *Let $A = (A, m, \eta, \Delta, \epsilon, S)$ be a color Hopf algebra and let M be a graded A-bimodule. Then there exists an isomorphism of graded spaces*

$$\mathrm{HH}^n_{\mathrm{gr}}(A, M) \simeq \mathrm{Ext}^n_{A\text{-gr}}(\mathbb{K}, {}^{ad} M), \quad n \geq 0,$$

where $\mathbb{K}$ is viewed as the trivial graded A-module via the counit ϵ, and ${}^{ad}M$ is the adjoint A-module associated to the graded A-bimodule M.

Proof. See [2]. □

Proposition 4.1 *Let L be a ϵ-Lie algebra and $\omega \in \mathrm{Z}_{gr}(L, \mathbb{K})$ a scalar 2-cocycle. Then the generalized enveloping algebra $U_\omega(L)$ of L is a color Hopf algebra.*

Proof. It 's shown in [2] that the graded tensor algebra $T(L)$ is a color Hopf algebra. Moreover it is easy to prove that the two-sided ideal generated by the elements, $v_1 \otimes v_2 - \epsilon(v_1, v_2) v_2 \otimes v_1 - [v_1, v_2] - \omega(v_1, v_2)$, where v_1, v_2 are homogeneous elements, is a graded Hopf ideal. It follows that the generalized enveloping algebra becomes a color algebra Hopf by quotient. □

Now we can apply the theorem for the generalized universal enveloping algebra $U_\omega(L)$ of a G-graded ε-Lie algebra L. In fact, if M is a graded $U_\omega(L)$-bimodule, the adjoint $U_\omega(L)$-module ${}^{ad}M$ is given by

$$ad(x)m = x.m - \varepsilon(|x|, |m|) m.x = [[x, m]] \tag{4.4}$$

for all homogeneous $x \in L$ and $m \in M$. Thus we have

Corollary 4.1 *Let L be a G-graded ε-Lie algebra and $U_\omega(L)$ its universal generalized enveloping algebra. Let M be a graded $U_\omega(L)$- bimodule. Then there exists a graded isomorphism*

$$\mathrm{HH}^n_{\mathrm{gr}}(U_\omega(L), M) = \mathrm{Ext}^n_{U_\omega(L)\text{-gr}}(\mathbb{K}, {}^{ad}M), \quad n \geq 0,$$

where ${}^{ad}M$ is the adjoint $U_\omega(L)$-module associated with the graded $U_\omega(L)$-bimodule M defined by (4.4).

It follows from [2] that the sequence

$$C : ... \to C_n \xrightarrow{d_n} C_{n-1} \to ...C_1 \xrightarrow{\epsilon} C_0 \tag{4.5}$$

is a G-graded $U(L_\omega)$-free resolution of the G-graded trivial $U(L_\omega)$-left module $\mathbb{K}$ via ϵ where $C_n = U(L_\omega) \otimes_{\mathbb{K}} \wedge^n_\varepsilon L_\omega$ and the operator d_n is given by

$$\begin{aligned} & d_n(u \otimes \langle x_1, \cdots, x_n \rangle) \\ &= \sum_{i=1}^{n} (-1)^{i+1} \varepsilon_i \; u x_i \otimes \langle x_1, \cdots, \hat{x_i}, \cdots, x_n \rangle \\ &\quad + \sum_{1 \leq i < j \leq n} (-1)^{i+j} \varepsilon_i \varepsilon_j \varepsilon(x_j, x_i) \; u \otimes \langle [x_i, x_j], x_1, \cdots, \hat{x_i}, \cdots, \hat{x_j}, \cdots, x_n \rangle , \end{aligned}$$

for all homogeneous elements $u \in U(L_\omega)$ and $x_i \in L$, with $\varepsilon_i = \prod_{h=1}^{i-1} \varepsilon(|x_h|, |x_i|)$ $i \geq 2$, $\varepsilon_1 = 1$ and the sign $\widehat{\ }$ indicates that the element below it must be omitted. The differential operator d maps $U(L_\omega) < y - 1 > \otimes_{\mathbb{K}} \wedge^n_\varepsilon L$ into itself, then it passes to the quotient, i.e., $\overline{d} : \overline{C}_n \to \overline{C}_{n-1}$ and satisfies that $\overline{d} \circ \overline{d} = 0$ where $\overline{C}_n = U_\omega(L) \otimes_{\mathbb{K}} \wedge^n_\varepsilon L$.

Proposition 4.2 *The sequence*

$$\overline{C} : ... \to \overline{C}_n \xrightarrow{\overline{d}_n} \overline{C}_{n-1} \to ...\overline{C}_1 \xrightarrow{\epsilon} C_0 \tag{4.6}$$

is a G-graded $U_\omega(L)$-free resolution of the G-graded trivial $U_\omega(L)$-left module $\mathbb{K}$ via ϵ.

Proof. Let $\{x_i\}_I$ be a homogeneous basis of L, where I is a well-ordered set. By Theorem 2.1 the elements

$$x_{k_1} \cdots x_{k_m} \otimes \langle x_{l_1} \cdots x_{l_n} \rangle \tag{4.7}$$

with

$$k_1 \leq \cdots \leq k_m \quad \text{and} \quad k_i < k_{i+1} \quad \text{if} \quad \varepsilon(|x_{k_i}|, |x_{k_i}|) = -1 \tag{4.8}$$

and

$$l_1 \leq \cdots \leq l_n \quad \text{and} \quad l_i < l_{i+1} \quad \text{if} \quad \varepsilon(|x_{l_i}|, |x_{l_i}|) = 1 \tag{4.9}$$

form a homogeneous basis of $\overline{C}_n$. The canonical filtration of $U_\omega(L)$, induces a filtration on the complex $\overline{C}$. The associated Z-graded complex $G(\overline{C})$ is G-graded and isomorphic to the $Z \times G$-graded complex $S(L) \otimes \wedge_\varepsilon L$. It follows from Lemma 3, [2], that the complex $G(\overline{C})$ is acyclic and consequently so is $\overline{C}$. □

Let M be a G-graded left (ω, L)-module, we define the n^{th} graded cohomology group of L with coefficients in M by

$$\mathrm{H}^n_{gr,\omega}(L, M) := \mathrm{Ext}^n_{U_\omega(L)-gr}(\mathbb{K}, M). \tag{4.10}$$

The modules on the right hand side can be computed using the left graded $U_\omega(L)$-projective resolution of $\mathbb{K}$. If M is a graded left (ω, L)-module, the graded cohomology groups are the graded homology groups of the complex:

$$\mathrm{Hom}_{U_\omega(L)-gr}(\overline{C}_n, M) = \mathrm{Hom}_{U_\omega(L)-gr}(U_\omega(L) \otimes \wedge^n_\varepsilon L, M) = \mathrm{Hom}_{\mathbb{K}-gr}(\wedge^n_\varepsilon L, M).$$

The coboundary operator in this cocomplex is

$$\overline{\delta}_n(f)\,(x_1, \cdots, x_{n+1}) \tag{4.11}$$

$$= \sum_{i=1}^{n+1} (-1)^{i+1} \varepsilon_i \; x_i f\,(x_1, \cdots, \hat{x_i}, \cdots, x_{n+1}) \tag{4.12}$$

$$+ \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \varepsilon_i \varepsilon_j \varepsilon(x_j, x_i) \; f\,([x_i, x_j], x_1, \cdots, \hat{x_i}, \cdots, \hat{x_j}, \cdots, x_{n+1})\,. \tag{4.13}$$

Theorem 4.2 *Let L be a G-graded ϵ-Lie algebra, $\omega \in \mathrm{H}^2_{gr}(L, \mathbb{K})$ and let $U_\omega(L)$ be its generalized universal enveloping algebra. Let M be a graded $U_\omega(L)$-bimodule. Let ${}^{ad}M$ be the adjoint graded left (L, ω)-module defined by*

$$ad(x)m = [[x, m]] := xm - \epsilon(|x|, |m|)mx$$

for all homogeneous elements $x \in L$ and $m \in M$. There exists an isomorphism

$$\mathrm{H}^n_{gr,\omega}(L, {}^{ad}M) \simeq \mathrm{HH}^n_{gr}(U_\omega(L), M), \quad n \geq 0. \tag{4.14}$$

Proof. It is a direct consequence from above and Corollary 4.1. □

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